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$$c'_1 = 0,\ d'_1 = 0;\ c'_2 = -\frac{d}{c},\ d'_2 = 0;\ c'_3 = \frac{be - cd}{c^2 - ba},\ d'_3 = \frac{d^2 - ec}{c^2 - bd};\ldots$$

The process will be materially simplified by any deficient terms, as when $d = 0$.

To find *the two greatest roots*, we have only to divide by the term containing the highest power of x so as to bring the equation into the following form, and then apply the last process:

$$1 + b\left(\frac{1}{x}\right) + c\left(\frac{1}{x^2}\right) + \ldots\ldots = 0.$$

THE APPLICATION OF THE EXPONENTIAL POLYGON TO THE HESSIAN.

BY PROF. H. T. EDDY, UNIV. OF CINCINNATI, CINCINNATI, OHIO.

1°. The second differential coefficient of any curve whose equation is

$$u = \psi(x, y) = 0$$

is

$$\frac{d^2y}{dx^2} = -\frac{\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dy}\right)^2 - 2\left(\frac{d^2u}{dxdy}\right)\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right) + \left(\frac{d^2u}{dy^2}\right)\left(\frac{du}{dx}\right)^2}{\left(\frac{du}{dy}\right)^3} \ldots\ldots (a)$$

(Todhunter's Differential Calculus, art. 180). The vanishing of (a) is the condition that the first differential coefficient is a maximum or a minimum. i. e. that the point of tangency is a point of inflection.

The condition then is

$$\left(\frac{d^2u}{dx^2}\right)\left(\frac{du}{dy}\right)^2 - 2\left(\frac{d^2u}{dxdy}\right)\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right) + \left(\frac{d^2u}{dy^2}\right)\left(\frac{du}{dx}\right)^2 = 0 \ldots\ldots (b)$$

which is a function of x and y and is the equation of the curve called the Hessian, which may be written $H = 0$. The Hessian passes through all points of inflection of $u = 0$.

2°. At a double point of the curve $u = 0$, the first differential coefficient is indeterminate

$$\therefore \qquad \frac{dy}{dx} = -\left(\frac{du}{dx}\right) \div \left(\frac{du}{dy}\right) = \frac{0}{0} \ldots\ldots\ldots\ldots (c)$$

Evaluate this vanishing fraction

$$\therefore \quad \frac{dy}{dx} = -\left\{\left(\frac{d^2u}{dxdy}\right)\frac{dy}{dx} + \left(\frac{d^2u}{dx^2}\right)\right\} \div \left\{\left(\frac{d^2u}{dy^2}\right)\frac{dy}{dx} + \left(\frac{d^2u}{dxdy}\right)\right\} \ldots (d)$$

(Todhunter's Differential Calculus, art. 190).

Eliminate $\frac{dy}{dx}$ by eq. (c) and clear of fractions and we again obtain $H=0$.
$\therefore$ the Hessian also passes through the double points of the curve $u=0$.

A triple point is the superposition of three double points and any multiple point of the order k is the superposition of k double points.

3° Let us move the origin of $u=0$ to a double point and make the two tangents at the point the coordinate axes. Then $u=0$ has no constant term or terms of the first degree, and no terms of the second degree except one containing xy, (Eddy's Anal. Geom. arts. 458, 453, 424), and may be written in the form $axy+t_{n>2}=0$, ... (e)
in which $t_{n>2}$ signifies "terms of higher degree than the second."

Now compute the Hessian of (e)

$$\left(\frac{du}{dx}\right)=ay+t_{n>1},\quad \left(\frac{du}{dy}\right)=ax+t_{n>1},\quad \left(\frac{d^2u}{dx^2}\right)=t_{n>0},$$
$$\left(\frac{d^2u}{dxdy}\right)=a+t_{n>0},\quad \left(\frac{d^2u}{dy^2}\right)=t_{n>0}.$$

$\therefore$ substituting, $H=2a^3xy+t_{n>2}=0$, which also has the axes both tangent to it for the same reason as eq. (e). Hence at a multiple point of $u=0$ the tangent of each branch is also tangent to a branch of the Hessian, and a multiple point of the order k in the curve $u=0$ is a multiple point of the same order in $H=0$.

4°. Let us move the origin of $u=0$ to a cusp and make the axis of x the double tangent at the cusp. Then there is no constant term in the equation, or terms of the first degree, and no term of the second degree except that containing y^2; as appears from considerations like those used in obtaining eq. (e). It may be written $ay^2+t_{n>2}=0$. . . . . (f)
Compute the Hessian of (f)

$$\left(\frac{du}{dx}\right)=t_{n>1},\quad \left(\frac{du}{dy}\right)=2ay+t_{n>1},\quad \left(\frac{d^2u}{dx^2}\right)=t_{n>0},$$
$$\left(\frac{d^2u}{dxdy}\right)=t_{n>0},\quad \left(\frac{d^2u}{dy^2}\right)=2a+t_{n>0}.$$

$\therefore$ $H=t_{n>2}=0$, which has in it no terms of the zero, first or second degrees, therefore it has three branches through the cusp. (Eddy's Anal. Geometry, art. 468). Hence at a cusp of $u=0$ there is a triple point of $H=0$, and two branches of the Hessian have a common tangent with the cusp.

5°. Similarly move the origin to a point of inflection and $u=0$ becomes, when the axis of x is tangent at the point,

$$ay + bxy + cy^2 + t_{n>2} = 0, \dots\dots\dots\dots (g)$$

which does not contain x^2. (Anal. Geom. art. 425). Compute the Hes. of (g),

$$\left(\frac{du}{dx}\right) = by + t_{n>1}, \quad \left(\frac{du}{dy}\right) = a + bx + 2cy + t_{n>1}, \quad \left(\frac{d^2u}{dx^2}\right) = t_{n>0},$$

$$\left(\frac{d^2u}{dxdy}\right) = 2c + t_{n>0}, \qquad \left(\frac{d^2u}{dy^2}\right) = b + t_{n>0}.$$

$\therefore H = t_{n>0} = 0$; and since there is no constant term the Hessian passes through the point of inflection as before stated.

This method may be applied in a similar manner to the singularities of higher orders which, as has been shown by Cayley, are merely the superposition or coincidence of several of these simple singularities.

BIPOLAR EQUATIONS - CARTESIAN OVALS.

BY PROF. WM. WOOLSEY JOHNSON, ST. JOHN'S COLLEGE, ANNAPOLIS, MD.

Every system of co-ordinates is specially adapted to the expression of some particular properties of those curves which admit of simple equations in that system. The object of this paper is to discuss the Bipolar system of point-coordinates and particularly the Cartesian Ovals, (the loci of Bipolar equations of the first degree).

Let the fixed points A_1 and A_2 be taken as poles, and denote by a the distance between the poles; then ρ_1 and ρ_2, denoting the distances of any point from the poles, are the bipolar coordinates of that point. We observe that any two points symmetrically situated with respect to the line A_1A_2 have the same pair of coordinates, hence the locus of every equation in bipolar coordinates is symmetrical with respect to the axis A_1A_2.

The equations of transformation to rectangular and to polar coordinates, taking A_1 for origin or pole and A_1A_2 for axis of x or initial line, are

$$\left.\begin{aligned} \rho_1^2 &= x^2 + y^2 = \rho^2 \\ \rho_2^2 &= x^2 + y^2 - 2ax + a^2 = \rho^2 - 2a\rho\cos\theta + a^2 \end{aligned}\right\} \dots\dots (1)$$

The values of ρ_1 and ρ_2 in terms of x and y being square roots, we observe that the rational rectangular equation obtained by transformation from any bipolar equation will always include the several loci which result from giving to ρ_1 and ρ_2 the ambiguous sign. Thus $\pm\rho_1 \pm \rho_2 = na \dots (2)$ will be included in a single rectangular equation. We shall therefore regard the results of different selections of the signs as the equations of